Global Well-posedness for the Keller-Segel system of equations in Critical Spaces

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Abstract

In this note, we prove that the Keller-Segel system of parabolic-parabolic type are globally well-posed for small initial data in $\dot{B}^{\frac{d}{p}-2}_{p,1} \times \dot{B}^{\frac{d}{p}}_{p,1}$, $1 \leq p < d$. Moreover, with additional assumptions on initial data, we prove the conservation of mass for the density function and nonnegativity of the density function and the concentration of the chemo-attractant.

1 Introduction

In this paper, we study the Keller-Segel system of parabolic-parabolic type in $\mathbb{R}^d$, $d \geq 2$. The system of equations is given by

$$(KS_\gamma) \begin{cases} u_t - \triangle u = -\nabla \cdot (u \nabla v) \\ v_t - \triangle v = u - \gamma v \end{cases}$$

where u is the density of cells and v is the concentration of the chemo-attractant. Keller-Segel [12] proposed a mathematical model describing the movement of biological cells in response to the chemical gradients. Its original form consists of four coupled reaction-advection-diffusion equations. By quasi-steady-state assumptions, these model can be reduced into a model for two functions u and v. The general form is

$$\begin{cases} u_t = \nabla\big(k_1(u,v)\nabla u - k_2(u,v) u \nabla v\big) + k_3(u,v) \\ v_t = D_v \triangle v + k_4(u,v) - k_5(u,v) v \end{cases}$$

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After the normalization, the minimal model can be obtained by taking $k_1 = k_2 = D_v = 1$, $k_3 = 0$, $k_4(u, v) = u$, and $k_5 = \gamma$. For the variations of this minimal model, see [9].

After the work of Childress-Percus [7], there is a vast body of literature dealing with quantitative behavior of solutions in bounded domain with proper boundary conditions. The main issue is whether solutions globally exist or blow up in finite time in terms of the critical mass and the spatial dimensions. An extensive review article by Horstmann [10] provides greater details. A further source of recent results can be found in the book of Perthame [19].

As a good approximation, people have studied the problem, so called, the parabolic-elliptic problem on a domain by considering the second equation to be $-\triangle v = u - \gamma v$.

$$(KSPE)\begin{cases} u_t = \triangle u - \nabla\cdot(u\nabla v), \ \text{ in } \Omega \\ 0 = \triangle v + u - \gamma v, \ \text{ in } \Omega \\ \nabla u\cdot n = 0, \ \ \nabla v\cdot n = 0, \ \text{ on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain and n is the normal vector on the boundary. By the second equation, one can reduce the system of equations into a single evolution equation of u. For the question of global existence versus finite breakdown for (KSPE), see [8, 11, 17, 18].

For the parabolic-parabolic problem on the whole spaces, we have a similar question: Can we find a critical space for the initial density such that it gives rise to global solutions of the system? Corrias and Perthame [4] first treated the case $d \geq 3$ and constructed a global weak solution for small data. Furthermore, in [5], they studied an asymptotic behavior of the weak solution for large time. For the problem on $\mathbb{R}^2$, see [1,6]. [1,4,5,6] deal with the problem with the density function in the Lebesgue space $L^{\frac{d}{2}}$. Recently, Kozono-Sugiyama [15] obtained the following result for the Keller-Segel system in Sobolev spaces, with $\gamma \geq 0$.

Theorem 1.1 *Let $d \geq 3$ and let $\max\{1, \frac{d}{4}\} < r < \frac{d}{2}$. There exists a constant $\epsilon_0(d, r) > 0$ such that if $u_0 \in H^{\frac{d}{r}-2,r}$ and $v_0 \in H^{\frac{d}{r},r}$ satisfy*

$$||(-\triangle)^{\frac{d}{2r}-1}u_0||_r + ||(\triangle)^{\frac{d}{2r}}||_r \leq \epsilon_0$$

then there exists a unique solution (u, v) such that

$$u \in C([0,\infty); H^{\frac{d}{r}-2}) \cap C((0,\infty); H^{2,r}) \cap C^1((0,\infty); L^r), \ \ v \in C([0,\infty); H^{\frac{d}{r}}) \cap C^1((0,\infty); L^r)$$

Moreover, such a solution has the following decay property;

$$||(\triangle)^{\sigma}u(t)||_r = O(t^{\frac{d}{2r}-1-\sigma}), \ \ \textit{for } \frac{d}{2r} - 1 \leq \sigma < 1, \ \ ||(\triangle)^{\zeta}u(t)||_r = O(t^{\frac{d}{2r}-\zeta}), \ \ \textit{for } \frac{d}{2r} \leq \zeta \leq \frac{d}{4r} + 1$$

They proved the above result by using Kato's method. If one takes the $L^p - L^q$ type estimates to the integral equations involving the heat semigroup operator, then the heat kernel generates time singularities near the origin. Then, one can define a function space with time weights to circumvent these time singularities.

The space of initial data in [1,4,5,6,15] are closely related to the scaling invariant class associated with (KS_γ), $\gamma = 0$. Let us call this system by (KS):

$$(KS)\begin{cases} u_t - \triangle u = \nabla\cdot(u\nabla v) \\ v_t - \triangle v = u \end{cases}$$

(KS) are invariant under a particular change of time and space scaling. Assume that (u, v) solves (KS). Then, the same is true for rescaled functions:

$$u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x), \quad v_\lambda(t,x) = v(\lambda^2 t, \lambda x) \tag{1}$$

Therefore, the Lebesgue spaces $L^{\frac{d}{2}} \times L^\infty$, which is closely related to spaces in [1,4,5,6], and the homogeneous potential spaces $\dot{H}^{\frac{d}{r}-2,r} \times \dot{H}^{\frac{d}{r},r}$ in [15] are scaling invariant spaces of the initial data.

In this paper, we study the system (KS) in scaling invariant Besov spaces $\dot{B}^{\frac{d}{p}-2}_{p,q} \times \dot{B}^{\frac{d}{p}}_{p,q}$. Since we need a L^∞ bound of solutions, we take $q = 1$. Motivated by the work of Chemin [3], we define a function space and its seminorm for u in the following way.

$$\mathscr{U} = \tilde{L}^\infty_t \dot{B}^{\frac{d}{p}-2}_{p,1} \cap \tilde{L}^2_t \dot{B}^{\frac{d}{p}-1}_{p,1} \cap \tilde{L}^1_t \dot{B}^{\frac{d}{p}}_{p,1}, \quad ||u||_{\mathscr{U}} = ||u||_{\tilde{L}^\infty_t \dot{B}^{\frac{d}{p}-2}_{p,1}} + ||u||_{\tilde{L}^2_t \dot{B}^{\frac{d}{p}-1}_{p,1}} + ||u||_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}}_{p,1}} \tag{2}$$

Similarly, we define a function space and its seminorm for v by

$$\mathscr{V} = \tilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1} \cap \tilde{L}^2_t \dot{B}^{\frac{d}{p}+1}_{p,1} \cap \tilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}, \quad ||v||_{\mathscr{V}} = ||v||_{\tilde{L}^\infty_t \dot{B}^{\frac{d}{p}}_{p,1}} + ||v||_{\tilde{L}^2_t \dot{B}^{\frac{d}{p}+1}_{p,1}} + ||v||_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}+2}_{p,1}} \tag{3}$$

The first result in this paper is the following.

Theorem 1.2 *Let $u_0 \in \dot{B}^{\frac{d}{p}-2}_{p,1}$ and $v_0 \in \dot{B}^{\frac{d}{p}}_{p,1}$, $1 \le p < d$. There exists a unique, global-in-time solution $(u,v) \in \mathscr{U} \times \mathscr{V}$ for sufficiently small data $(u_0, v_0) \in \dot{B}^{\frac{d}{p}-2}_{p,1} \times \dot{B}^{\frac{d}{p}}_{p,1}$ such that*

$$u \in C([0,\infty); \dot{B}^{\frac{d}{p}-2}_{p,1}), \quad v \in C([0,\infty); \dot{B}^{\frac{d}{p}}_{p,1})$$

But, from biological point of view, u and v should be nonnegative. Moreover, the mass must be conserved because there is no external forces acting on the system. With additional assumptions on initial data, we can prove the following two Theorems. The same results are also addressed in [15].

Theorem 1.3 *Let $u_0 \in \dot{B}^{\frac{d}{p}-2}_{p,1} \cap L^1$ and $v_0 \in \dot{B}^{\frac{d}{p}}_{p,1} \cap L^1$, $1 \le p < d$. If initial data is small enough* *in* $\dot{B}^{\frac{d}{p}-2}_{p,q} \times \dot{B}^{\frac{d}{p}}_{p,q}$*, then, the solution* u *in Theorem 1.2 satisfies the mass conservation.*

Theorem 1.4 *Let* $u_0 \in \dot{B}^{\frac{d}{p}-2}_{p,1} \cap L^1 \cap L^2$ *and* $v_0 \in \dot{B}^{\frac{d}{p}}_{p,1} \cap L^1$*,* $1 \le p < d$*. If initial data is small enough in* $\dot{B}^{\frac{d}{p}-2}_{p,q} \times \dot{B}^{\frac{d}{p}}_{p,q}$*, then, the nonnegativity for the solution* (u, v) *holds.*

Remark 1 *Our result is an improvement of [15] in the following sense.*
(a) We do not need a lower bound of p*. We only need* $1 \le p < d$*.*
(b) $d = 2$ *is allowed.*
(c) u *can be chosen in spaces with negative regularities.*

This paper consists as follows. In chapter 2, we present a short introduction to the Littlewood-Paley theory. For more results of this subject, see [2,3]. In chapter 3, we prove Theorem 1.2. First, we obtain a priori estimate. Then we prove the existence and uniqueness by iterating the system using the a priori estimate. We also prove the continuity of the solution in time up to $t = 0$. In chapter 4, we prove Theorem 1.3 and 1.4.

In this paper, $A \lesssim B$ means there exists a universal constant C (which depends on the contexts) such that $A \le CB$.

2 Preliminaries

2.1 Littlewood-Paley Decomposition

We take a couple of smooth functions (χ, ϕ) supported on $\{\xi; |\xi| \le 1\}$ with values in $[0, 1]$ such that for all $\xi \in R^d$,

$$\chi(\xi) + \sum_{j=0}^{\infty} \psi(2^{-j}\xi) = 1$$

where $\psi(\xi) = \phi(\frac{\xi}{2}) - \phi(\xi)$. We denote $\psi(2^{-j}\xi)$ by $\psi_j(\xi)$. This is called the Littlewood-Paley dyadic partition of unity. We apply this decomposition to elements in $\mathscr{S}'$. Let

$$\triangle_j u = 0 \quad \text{if} \quad j \le -2 \quad \triangle_{-1} u = \chi(D)u = \tilde{h} \star u \quad \text{with} \quad \tilde{h} = \mathscr{F}^{-1}\chi$$

$$\triangle_j u = \psi_j(D)u = 2^{jd} \int h(2^j y)u(x - y)dy \quad \text{with} \quad h = \mathscr{F}^{-1}\psi, \quad \text{if} \quad j \ge 0$$

We define a nonhomogeneous Littlewood-Paley decomposition as follows.

$$u = \sum_{j=-1}^{\infty} \triangle_j u, \quad u \in \mathscr{S}'$$

We introduce the following low frequency cut-off: $S_j u = \sum_{l \leq j-1} \triangle_l u$. Then, $u = \lim_{j \to \infty} S_j u$ in $\mathscr{S}'$. Conversely, let $\tilde{C}$ be a ring. Suppose that the support of $\mathscr{F} u_j$ is contained in the annulus $2^j \tilde{C}$ and $||u_j||_{L^\infty} \lesssim 2^{jd}$. Then, $\sum_{j \geq -1} u_j < \infty$ in $\mathscr{S}'$.

We also define a homogeneous Littlewood-Paley decomposition. We take a couple of smooth functions (χ, ϕ) as before. We define homogeneous dyadic blocks and low frequency cut-offs by

$$\dot{\triangle} u = \psi(2^{-j} D) u \quad \text{for all} \quad j \in \mathbb{Z}, \quad \dot{S}_j u = \chi(2^{-j} D) u \quad \text{for all} \quad j \in \mathbb{Z}$$

Then, for $u \in \mathscr{S}'$, we have $u = \sum_{j \in \mathbb{Z}} \dot{\triangle}_j u$ modulo a polynomial only.

2.2 Besov Spaces

Let $s \in R$, $p, q \in [1, \infty]$. Then the inhomogeneous and homogeneous Besov semi-norms are defined, respectively, by

$$||u||_{B^s_{p,q}} = ||S_0 u||_{L^p} + (\sum_{j \geq -1} 2^{qjs} ||\triangle_j u||^q_{L^P})^{\frac{1}{q}}, \ \ ||u||_{\dot{B}^s_{p,q}} = (\sum_{j \in \mathbb{Z}} 2^{qjs} ||\triangle_j u||^q_{L^P})^{\frac{1}{q}}$$

If u is time-dependent,

$$||u||_{L^\rho_t B^s_{p,q}} = ||S_0 u||_{L^\rho_t L^p} + ||(\sum_{j \geq -1} 2^{qjs} ||\triangle_j u||^q_{L^p})^{\frac{1}{q}}||_{L^\rho_t}, \ \ ||u||_{L^\rho_t \dot{B}^s_{p,q}} = ||(\sum_{j \in \mathbb{Z}} 2^{qjs} ||\triangle_j u||^q_{L^p})^{\frac{1}{q}}||_{L^\rho_t}$$

By changing the order of time integration and the summation, we define semi-norms as

$$||u||_{\tilde{L}^\rho_t B^s_{p,q}} = ||S_0 u||_{L^\rho_t L^p} + (\sum_{j \geq -1} 2^{qjs} ||\triangle_j u||^q_{L^\rho_t L^p})^{\frac{1}{q}}, \ \ ||u||_{\tilde{L}^\rho_t \dot{B}^s_{p,q}} = (\sum_{j \in \mathbb{Z}} 2^{qjs} ||\triangle_j u||^q_{L^\rho_t L^p})^{\frac{1}{q}}$$

By changing these orders, one can avoid the time singularity at the origin of the heat kernel when we estimate the solution in the integral form in chapter 3. According to Minkowski inequality, we have

$$||u||_{\tilde{L}^\rho_t B^s_{p,q}} \leq ||u||_{L^\rho_t B^s_{p,q}} \quad \text{if} \quad \rho \leq q, \quad ||u||_{\tilde{L}^\rho_t B^s_{p,q}} \geq ||u||_{L^\rho_t B^s_{p,q}} \quad \text{if} \quad \rho \geq q$$

The interest in decomposing a tempered distribution into a sum of dyadic blocks $\triangle_j u$, whose support in Fourier space is localized in a corona, comes from the nice behavior of these blocks with respect to differentiations. This fact is illustrated by the following Bernstein's lemma. For its proof, see [2].

Lemma 2.1 *(a) Let $1 \leq p \leq q \leq \infty$, $k \in \mathbb{N}$, and $R > 0$. For $f \in \mathscr{S}'$ whose Fourier transform*

$\hat{f}$ *is supported in the ball* $|\xi| \lesssim \lambda R$*,*

$$\sup_{|\alpha|=k} ||\partial^\alpha f||_{L^p} \lesssim \lambda^k ||f||_{L^p}, \quad ||f||_{L^q} \lesssim \lambda^{d(\frac{1}{p}-\frac{1}{q})} ||f||_{L^p}$$

(b) For $f \in \mathscr{S}'$ *whose Fourier transform* $\hat{f}$ *is supported in the corona* $|\xi| \sim \lambda R$*,*

$$\sup_{|\alpha|=k} ||\partial^\alpha f||_{L^p} \simeq \lambda^k ||f||_{L^p}, \quad ||f||_{L^q} \lesssim \lambda^{d(\frac{1}{p}-\frac{1}{q})} ||f||_{L^p}$$

From this Lemma, we can prove the following two embedding properties, which we will use in chapter 3 and chapter 4.

$$\dot{B}^{\frac{d}{p}}_{p,1} \subset L^\infty, \quad \dot{B}^{\frac{d}{p}-1}_{p,1} \subset L^d \text{ for } p < d$$

The fundamental idea of the paper [3] is to localize the heat equation and estimate each dyadic block in $L^\rho_t L^p_x$. In this way, one can extract maximal regularities in L^1 in time from the heat kernel.

Lemma 2.2 *Let* $\mathscr{C}$ *be a ring. There exists a positive constant* C *such that for any* $p \in [1, \infty]$*, for any couple* (t, λ) *of positive real numbers, we have*

$$||e^{t\triangle} u||_{L^p} \le C e^{-t(\lambda)^2} ||u||_{L^p} \text{ for supp } \hat{u} \in \lambda\mathscr{C}$$

2.3 Paraproduct

The concept of paraproduct is to deal with the interaction of two functions in terms of low or high frequency parts. For u, v two tempered distributions, we have the formal decomposition:

$$uv = \sum_{i,j} \triangle_i u \triangle_j v$$

The idea of paradifferential calculus is to split uv into three parts. The fist part, denoted by $T_u v$ and called paraproduct of v by u, corresponds to terms $\triangle_i u \triangle_j v$ where i is small compared with j. By the symmetry, the second part is $T_v u$. The last part, the remainder, corresponds to the dyadic blocks with comparable frequencies.

Definition 2.3 *Let* u *and* v *be two tempered distributions. Then,*

$$uv = T_u v + T_v u + R(u, v)$$

$$T_u v = \sum_{i \le j-2} \triangle_i u \triangle_j v = \sum_{j \in \mathbb{Z}} S_{j-1} u \triangle_j v, \quad R(u, v) = \sum_{|j-j'| \le 1} \triangle_j u \triangle_{j'} v$$

The paraproduct of two tempered distributions is always defined because terms are localized by dyadic pieces. The regularity of $T_u v$ is mainly determined by v. The remainder may not be defined. Roughly, it is defined when u and v belong to function spaces whose sum of regularity index is positive. We list continuity properties for the inhomogeneous paraproduct and the remainder. The reader is referred to [2] for more results on the subject.

Proposition 2.4 *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$.*
(i) T is a bilinear continuous operator from $L^\infty \times B^s_{p,q}$ to $B^s_{p,q}$ such that

$$||T||_{\mathscr{L}(L^\infty \times B^s_{p,q} \to B^s_{p,q})} \lesssim C^{|s|+1}$$

(ii) Let $(s_1, s_2) \in \mathbb{R}^2$ and $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$. Assume that

$$\tfrac{1}{p} \le \tfrac{1}{p_1} + \tfrac{1}{p_2} \le 1, \quad \tfrac{1}{q} \le \tfrac{1}{q_1} + \tfrac{1}{q_2}, \quad s_1 + s_2 > 0$$

Then, the remainder R maps $B^{s_1}_{p_1,q_1} \times B^{s_2}_{p_2,q_2}$ to $B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,q}$ such that

$$||R(u,v)||_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,q}} \lesssim \frac{C^{|s_1+s_2|+1}}{s_1+s_2} ||u||_{B^{s_1}_{p_1,q_1}} ||v||_{B^{s_2}_{p_2,q_2}}$$

(iii) For $s > 0$, $B^s_{p,q} \cap L^\infty$ is an algebra and

$$||uv||_{B^s_{p,q}} \lesssim ||u||_{L^\infty} ||v||_{B^s_{p,q}} + ||v||_{L^\infty} ||u||_{B^s_{p,q}}$$

2.4 Two more lemmas

Finally, we need two additional Lemmas in chapter 4. If one takes L^p norm to the fluctuation term in the integral expression of the solution, the the heat kernel generates time singularities near the origin (Lemma 2.5). Then, we control these time singularities by using the Hardy-Littlewood-Sobolev inequality (Lemma 2.6).

Lemma 2.5 *The Heat Kernel Estimate: For all $f \in L^p$,*

$$||\nabla e^{t\triangle} f||_{L^r} \lesssim t^{-\frac{1}{2}-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})} ||f||_{L^p}, \quad 1 \le p \le r \le \infty$$

Lemma 2.6 *Hardy-Littlewood-Sobolev Inequality [16]: Let $0 < \lambda < d$, $\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{q} = 2$. Then,*

$$\Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \frac{1}{|x-y|^\lambda} g(y) dy dx \Big| \lesssim ||f||_{L^p} ||g||_{L^q}$$

In particular, for the one dimensional case,

$$\Big| \int_0^t \frac{1}{|t-s|^{\frac{1}{2}}} a(s) ds \Big| \lesssim ||a||_{L^2}$$

3 Proof of Theorem 1.2

This chapter consists of three parts. First, we obtain a priori estimate under the assumption that there exists a global-in-time smooth solution to (KS). Then, from the a priori estimate, we can prove the existence and the uniqueness of the solution by iterating the system. Finally, we prove that the solution is continuous in time with values in the same space of initial data.

3.1 A priori Estimate

We construct a solution in the integral form.

$$u(t) = e^{t\triangle}u_0 + \int_0^t e^{(t-s)\triangle}\nabla\cdot(u\nabla v)(s)ds, \quad v(t) = e^{t\triangle}v_0 + \int_0^t e^{(t-s)\triangle}u(s)ds \tag{4}$$

We apply the Fourier-localized operator $\triangle_j$ to the equation of u. Then,

$$\triangle_j u(t) = e^{t\triangle}\triangle_j u_0 + \int_0^t e^{(t-s)\triangle}\triangle_j\nabla\cdot(u\nabla v)(s)ds \tag{5}$$

We take the L^p norm in the spatial variables. By Lemma 2.2,

$$||\triangle_j u(t)||_{L^p} \lesssim e^{-t2^{2j}}||\triangle_j u_0||_{L^p} + \int_0^t e^{-(t-s)2^{2j}}2^j||\triangle_j(u\nabla v)(s)||_{L^p}ds \tag{6}$$

We take the L^∞ norm in time. By Young's inequality in time,

$$||\triangle_j u(t)||_{L_t^\infty L^p} \lesssim ||\triangle_j u_0||_{L^p} + 2^j||\triangle_j(u\nabla v)||_{L_t^1 L^p} \tag{7}$$

We multiply (6) by $2^{j(\frac{d}{p}-2)}$ and add them up. Then, we obtain

$$||u||_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}-2}} \lesssim ||u_0||_{\dot{B}_{p,1}^{\frac{d}{p}-2}} + ||u\nabla v||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}-1}} \tag{8}$$

Similarly, by taking L^1 in time to (6),

$$||u||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}}} \lesssim ||u_0||_{\dot{B}_{p,1}^{\frac{d}{p}-2}} + ||u\nabla v||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}-1}} \tag{9}$$

By adding (8) and (9),

$$||u||_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}-2}} + ||u||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}}} \lesssim ||u_0||_{\dot{B}_{p,1}^{\frac{d}{p}-2}} + ||u\nabla v||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}-1}} \tag{10}$$

By interpolating two terms in the left-hand side,

$$||u||_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}-2}} + ||u||_{\tilde{L}_t^2 \dot{B}_{p,1}^{\frac{d}{p}-1}} + ||u||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}}} \lesssim ||u_0||_{\dot{B}_{p,1}^{\frac{d}{p}-2}} + ||u\nabla v||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}-1}} \tag{11}$$

Let us do the same calculation to v.

$$||v||_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}}} + ||v||_{\tilde{L}_t^2 \dot{B}_{p,1}^{\frac{d}{p}+1}} + ||v||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}+2}} \lesssim ||v_0||_{\dot{B}_{p,1}^{\frac{d}{p}}} + ||u_0||_{\dot{B}_{p,1}^{\frac{d}{p}-2}} + ||u||_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}-2}} + ||u||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}}} \tag{12}$$

Therefore, from (11) and (12),

$$||u||_{\mathscr{U}} + ||v||_{\mathscr{V}} \lesssim ||u_0||_{\dot{B}_{p,1}^{\frac{d}{p}-2}} + ||v_0||_{\dot{B}_{p,1}^{\frac{d}{p}}} + ||u\nabla v||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}-1}} \tag{13}$$

It remains to estimate the nonlinear term $(u\nabla v)$. We decompose $(u\nabla v)$ as a paraproduct.

$$u\nabla v = T_u \nabla v + T_{\nabla v} u + R(u, \nabla v) \tag{14}$$

We take the operator $\triangle_j$ to $(u\nabla v)$. Up to finitely many terms,

$$\triangle_j(u\nabla v) = S_j u \triangle_j \nabla v + S_j \nabla v \triangle_j u + \sum_{k>j} \triangle_k u \triangle_k (\nabla v) = (I) + (II) + (III)$$

First, we take $L_t^1 L^p$ norm to (I).

$$||S_j u \triangle_j \nabla v||_{L_t^1 L^p} \leq ||S_j u||_{L_t^1 L^\infty} ||\triangle_j \nabla v||_{L_t^\infty L^p} \lesssim ||u||_{L_t^1 L^\infty} ||\triangle_j \nabla v||_{L_t^\infty L^p} \tag{15}$$

Multiplying by $2^{j(\frac{d}{p}-1)}$ and adding them up,

$$||T_u \nabla v||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}-1}} \lesssim ||u||_{L_t^1 L^\infty} \times ||\nabla v||_{\tilde{L}_t^\infty \dot{B}_{p,1}^{\frac{d}{p}-1}} \tag{16}$$

Next, we deal with (II).

$$||S_j(\nabla v)\triangle_j u||_{L_t^1 L^p} \lesssim ||S_j \nabla v||_{L_t^2 L^\infty} ||\triangle_j u||_{L_t^2 L^p} \tag{17}$$

Then,

$$||T_{\nabla v} u||_{\tilde{L}_t^1 \dot{B}_{p,1}^{\frac{d}{p}-1}} \lesssim ||\nabla v||_{L_t^2 L^\infty} \times ||u||_{\tilde{L}_t^2 \dot{B}_{p,1}^{\frac{d}{p}-1}} \tag{18}$$

Finally, we estimate the remainder.

$$
\begin{aligned}
& ||\sum_{k>j} \triangle_k u \triangle_k(\nabla v)||_{L^1_t L^p} \lesssim \sum_{k>j} ||\triangle_k u \triangle_k(\nabla v)||_{L^1_t L^p} \lesssim \sum_{k>j} ||\triangle_k u||_{L^\infty_t L^{2p}} ||\triangle_k(\nabla v)||_{L^1_t L^{2p}} \\
\lesssim \quad & \sum_{k>j} 2^{kd(\frac{1}{p}-\frac{1}{2p})} ||\triangle_k u||_{L^\infty_t L^p} \times 2^{kd(\frac{1}{p}-\frac{1}{2p})} ||\triangle_k(\nabla v)||_{L^1_t L^p} \\
= \quad & \sum_{k>j} 2^{k\frac{d}{p}} \times 2^{k(\frac{d}{p}-2)} ||\triangle_k u||_{L^\infty_t L^p} \times 2^{k(\frac{d}{p}+1)} ||\triangle_k(\nabla v)||_{L^1_t L^p} \times 2^{-k(\frac{d}{p}-2)} \times 2^{-k(\frac{d}{p}+1)} \\
= \quad & \sum_{k>j} 2^{-k(\frac{d}{p}-1)} \times 2^{k(\frac{d}{p}-2)} ||\triangle_k u||_{L^\infty_t L^p} \times 2^{k(\frac{d}{p}+1)} ||\triangle_k(\nabla v)||_{L^1_t L^p} \qquad (19)
\end{aligned}
$$

We multiply by $2^{j(\frac{d}{p}-1)}$ to (19).

$$
\begin{aligned}
& 2^{j(\frac{d}{p}-1)} \sum_{k>j} 2^{-k(\frac{d}{p}-1)} \times 2^{k(\frac{d}{p}-2)} ||\triangle_k u||_{L^\infty_t L^p} \times 2^{k(\frac{d}{p}+1)} ||\triangle_k(\nabla v)||_{L^1_t L^{2p}} \\
= \quad & \sum_{k>j} 2^{(j-k)(\frac{d}{p}-1)} \times 2^{k(\frac{d}{p}-2)} ||\triangle_k u||_{L^\infty_t L^p} \times 2^{k(\frac{d}{p}+1)} ||\triangle_k(\nabla v)||_{L^1_t L^{2p}} \qquad (20)
\end{aligned}
$$

Since $p < d$, $\frac{d}{p} - 1 > 0$. By applying Young's inequality to (20),

$$
||R(u, \nabla v)||_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}-1}_{p,1}} \lesssim ||u||_{\tilde{L}^\infty_t \dot{B}^{\frac{d}{p}-2}_{p,1}} \times ||\nabla v||_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}+1}_{p,1}} \qquad (21)
$$

By (16), (18), and (21),

$$
\begin{aligned}
||u \nabla v||_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}-1}_{p,1}} \quad \lesssim \quad & ||u||_{L^1_t L^\infty} \cdot ||\nabla v||_{\tilde{L}^\infty_t \dot{B}^{\frac{d}{p}-1}_{p,1}} + ||\nabla v||_{L^2_t L^\infty} \cdot ||u||_{\tilde{L}^2_t \dot{B}^{\frac{d}{p}-1}_{p,1}} \\
+ \quad & ||u||_{\tilde{L}^\infty_t \dot{B}^{\frac{d}{p}-2}_{p,1}} \cdot ||\nabla v||_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}+1}_{p,1}} \qquad (22)
\end{aligned}
$$

By Minkowski inequality and the embedding property $\dot{B}^{\frac{d}{p}}_{p,1} \subset L^\infty$,

$$
||u||_{L^1_t L^\infty} \lesssim ||u||_{\tilde{L}^1_t \dot{B}^{\frac{d}{p}}_{p,1}}, \quad ||\nabla v||_{L^2_t L^\infty} \lesssim ||\nabla v||_{\tilde{L}^2_t \dot{B}^{\frac{d}{p}}_{p,1}} \qquad (23)
$$

In sum,

$$
||u||_{\mathscr{U}} + ||v||_{\mathscr{V}} \lesssim ||u_0||_{\dot{B}^{\frac{d}{p}-2}_{p,1}} + ||v_0||_{\dot{B}^{\frac{d}{p}}_{p,1}} + ||u||_{\mathscr{U}} \times ||v||_{\mathscr{V}} \qquad (24)
$$

3.2 Existence and Uniqueness

We consider the following iteration scheme:

$$
u^{(1)}(t,x) = e^{t\triangle} u_0, \quad v^{(1)}(t,x) = e^{t\triangle} v_0
$$

$$u^{(m+1)}(t) = u^{(1)}(t) + \int_0^t e^{(t-s)\triangle}\nabla\cdot(u^{(m)}\nabla v^{(m)})(s)ds$$

$$v^{(m+1)}(t) = v^{(1)}(t) + \int_0^t e^{(t-s)\triangle}u^{(m+1)}(s)ds$$

In [15], They defined the iteration scheme such that $v^{(m+1)}$ is defined in terms of $u^{(m)}$. This requires a new algorithm to make the iteration closed. By defining $v^{(m+1)}$ in terms of $u^{(m+1)}$, however, we can treat the iteration process as a usual one. We solve by $u^{(2)}$ using $u^{(1)}$ and $v^{(1)}$ first. Then, we solve $v^{(2)}$ by $u^{(2)}$, so on.

$$u^{(1)}, v^{(1)} \to u^{(2)} \to v^{(2)} \to u^{(3)} \to v^{(3)} \to \cdots$$

Then, by the a priori estimate (24),

$$||u^{(m+1)}||_{\mathscr{U}} \lesssim ||u_0||_{\dot{B}^{\frac{d}{p}-2}_{p,1}} + ||u^{(m)}||_{\mathscr{U}} \times ||v^{(m)}||_{\mathscr{V}} \tag{25}$$

$$||v^{(m+1)}||_{\mathscr{V}} \lesssim ||u_0||_{\dot{B}^{\frac{d}{p}-2}_{p,1}} + ||v_0||_{\dot{B}^{\frac{d}{p}}_{p,1}} + ||u^{(m)}||_{\mathscr{U}} \times ||v^{(m)}||_{\mathscr{V}} \tag{26}$$

These two estimates imply uniform bounds of $\{(u^{(m)}, v^{(m)})\}$ in $\mathscr{U}\times\mathscr{V}$ for sufficiently small initial data in $\dot{B}^{\frac{d}{p}-2}_{p,1} \times \dot{B}^{\frac{d}{p}}_{p,1}$. To prove the existence of a solution to (KS), we need to show that there exists a limit (u, v) of $\{(u^{(m)}, v^{(m)})\}$ solving (KS). In order that, we estimate $(u^{(m+1)} - u^{(m)})$ and $(v^{(m+1)} - v^{(m)})$. By definition of $u^{(m)}$ and $v^{(m)}$,

$$u^{(m+1)}(t) - u^{(m)}(t) = \int_0^t e^{(t-s)\triangle}\nabla\cdot\big(u^{(m)}\nabla v^{(m)} - u^{(m-1)}\nabla v^{(m-1)}\big)(s)ds \tag{27}$$

$$v^{(m+1)}(t) - v^{(m)}(t) = \int_0^t e^{(t-s)\triangle}\big(u^{(m+1)}(s) - u^{(m)}(s)\big)ds \tag{28}$$

Then,

$$\begin{aligned}
||u^{(m+1)} - u^{(m)}||_{\mathscr{U}} &\lesssim ||(u^{(m)}\nabla v^{(m)} - u^{(m-1)}\nabla v^{(m-1)}||_{\tilde{L}^1_t\dot{B}^{\frac{d}{p}-1}_{p,1}} \\
&= ||u^{(m)}\big(\nabla v^{(m)} - \nabla v^{(m-1)}\big) + \big(u^{(m)} - u^{(m-1)}\big)\nabla v^{(m-1)}||_{\tilde{L}^1_t\dot{B}^{\frac{d}{p}-1}_{p,1}} \\
&\lesssim ||u^{(m)}||_{\mathscr{U}}||v^{(m)} - v^{(m-1)}||_{\mathscr{V}} + ||u^{(m)} - u^{(m-1)}||_{\mathscr{U}}||v^{(m-1)}||_{\mathscr{V}} \\
&\lesssim \frac{1}{4}||v^{(m)} - v^{(m-1)}||_{\mathscr{V}} + \frac{1}{4}||u^{(m)} - u^{(m-1)}||_{\mathscr{U}}
\end{aligned} \tag{29}$$

where we use the uniform bound of $\{(u^{(m)}, v^{(m)})\}$. For sufficiently small data, we can assume that these uniform bound is less than $\frac{1}{4}$. Similarly,

$$||v^{(m+1)} - v^{(m)}||_{\mathscr{V}} \lesssim ||u^{(m)}||_{\mathscr{U}}||v^{(m)} - v^{(m-1)}||_{\mathscr{V}} + ||u^{(m)} - u^{(m-1)}||_{\mathscr{U}}||v^{(m-1)}||_{\mathscr{V}}$$
$$\lesssim \frac{1}{4}||v^{(m)} - v^{(m-1)}||_{\mathscr{V}} + \frac{1}{4}||u^{(m)} - u^{(m-1)}||_{\mathscr{U}} \tag{30}$$

Therefore, $\{u^{(m)}, v^{(m)}\}$ is a Cauchy sequence in $\mathscr{U} \times \mathscr{V}$ and its limit defines a solution to (KS).

Next, we prove the uniqueness. Suppose there are two solutions (u_1, v_1), (u_2, v_2). Then,

$$u_1(t) - u_2(t) = \int_0^t e^{(t-s)\triangle}\nabla \cdot \big(u_1\nabla v_1 - u_2\nabla v_2\big)(s)ds, \quad v_1(t) - v_2(t) = \int_0^t e^{(t-s)\triangle}\big(u_1 - u_2\big)(s)ds \tag{31}$$

It is easy to show that

$$||u_1 - u_2||_{\mathscr{U}} \lesssim ||u_2||_{\mathscr{U}}||v_1 - v_2||_{\mathscr{V}} + ||u_1 - u_2||_{\mathscr{U}}||v_1||_{\mathscr{V}} \tag{32}$$

Similarly,

$$||v_2 - v_1||_{\mathscr{V}} \lesssim ||u_2||_{\mathscr{U}}||v_1 - v_2||_{\mathscr{V}} + ||u_1 - u_2||_{\mathscr{U}}||v_1||_{\mathscr{V}} \tag{33}$$

Therefore, for sufficiently small solutions,

$$||u_1 - u_2||_{\mathscr{U}} + ||v_2 - v_1||_{\mathscr{V}} \leq \frac{1}{2}\big(||u_1 - u_2||_{\mathscr{U}} + ||v_2 - v_1||_{\mathscr{V}}\big) \tag{34}$$

which implies the uniqueness of the solution.

3.3 Continuity in time

Finally, we show that the solution (u, v) is continuous in time with values in $\dot{B}^{\frac{d}{p}-2}_{p,q} \times \dot{B}^{\frac{d}{p}}_{p,q}$. We consider the difference between the solution and its initial data. We only prove the continuity in time for u. The proof for v is the same.

$$u(t) - u_0 = \big(e^{t\triangle} - 1\big)u_0 + \int_0^t e^{(t-s)\triangle}\nabla \cdot (u\nabla v)(s)ds \tag{35}$$

By the Minkowski inequality, for $t \in [0, T]$,

$$||u(t) - u_0||_{\dot{B}^{\frac{d}{p}-2}_{p,q}} \lesssim \sum_{j\in\mathbb{Z}}(1 - e^{-t2^{2j}})2^{j(\frac{d}{p}-2)}||\triangle_j u_0||_{L^p} + ||u\nabla v||_{\tilde{L}^1_t(0,T;\dot{B}^{\frac{d}{p}-1}_{p,1})} \tag{36}$$

Let us estimate the first term in the right-hand side. Since its sum is convergent, there exists a large number N such that

$$\sum_{|j|\geq N}(1-e^{-t2^{2j}})2^{j(\frac{d}{p}-2)}||\triangle_j u_0||_{L^p} < \epsilon \tag{37}$$

uniformly in time. Since

$$\lim_{t\to 0}\sum_{|j|<N}(1-e^{-t2^{2j}})2^{j(\frac{d}{p}-2)}||\triangle_j u_0||_{L^p} = 0 \tag{38}$$

the first term in the right-hand side of (36) goes to 0 as $T\to 0$. By the same argument, the last term in the right-hand side of (35) goes to zero as $T\to 0$. Therefore,

$$\lim_{t\to 0}||u(t)-u_0||_{\dot{B}^{\frac{d}{p}-2}_{p,q}} = 0 \tag{39}$$

which means that the solution is continuous in time up to $t=0$. By translating in time, we can prove the continuity in time for any $t<\infty$. This completes the proof of theorem 1.2. ■

4 Proof of Theorem 1.3, Theorem 1.4

In this chapter, we show that the mass of u is conserved:

$$\int u(t,x)dx = \int u_0(x)dx \tag{40}$$

with additional condition for u_0 : $u_0 \in \dot{B}^{\frac{d}{p}-2}_{p,1}\cap L^1$. Moreover, the solution is nonnegative for almost all $t>0$ and $x\in\mathbb{R}^d$ with additional condition on u_0 : $u_0\in \dot{B}^{\frac{d}{p}-2}_{p,1}\cap L^1\cap L^2$.

4.1 Proof of Theorem 1.3

First, we show that L^1 norm of u in Theorem 1.2 does not blow up in finite time.

$$\begin{aligned} ||u(t)||_{L^1} &\lesssim ||u_0||_{L^1} + \int_0^t (t-s)^{-\frac{1}{2}}||(u\nabla v)(s)||_{L^1}ds \\ &\lesssim ||u_0||_{L^1} + ||u||_{L^\infty_t L^1}\int_0^t (t-s)^{-\frac{1}{2}}||\nabla v(s)||_{L^\infty}ds \end{aligned} \tag{41}$$

Since $\nabla v\in L^2_tL^\infty$, by the Hardy-Littlewood-Sobolev inequality, we have

$$||u(t)||_{L^1}\lesssim ||u_0||_{L^1} + ||u||_{L^\infty_t L^1}\times ||v||_{\mathscr{V}} \tag{42}$$

Therefore, by Theorem 1.2, for small data in $\dot{B}^{\frac{d}{p}-2}_{p,1} \times \dot{B}^{\frac{d}{p}}_{p,1}$,

$$||u||_{L^\infty_t L^1} \lesssim ||u_0||_{L^1} \tag{43}$$

Now, we prove the equality (40) by applying the same idea used in [15] to our problem. Let us take a smooth, compactly supported function ϕ such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| > 2$. We define a series of functions ϕ_m's by setting $\phi_m(x) = \phi(\frac{x}{m})$, $m = 1, 2, \cdots$. Then, for each m, we have ϕ_m supported on $\{x; |x| \leq 2m\}$, with $||\nabla^j \phi_m||_{L^\infty} \lesssim m^{-j}$. We test u over ϕ_m's.

$$\begin{aligned}
\Big|\frac{d}{dt}\int u(t,x)\phi_m(x)dx\Big| &= \Big|\int \triangle u(t,x)\phi_m(x)dx - \int \nabla\cdot(u\nabla v)(t,x)\phi_m(x)dx\Big| \\
&= \Big|\int u(t,x)\triangle\phi_m(x)dx + \int (u\nabla v)(t,x)\cdot\nabla\phi_m(x)dx\Big| \\
&\leq ||u||_{L^\infty_t L^1}||\triangle\phi_m||_{L^\infty} + ||u||_{L^\infty_t L^1}||\nabla v(t)||_{L^\infty}||\nabla\phi_m||_{L^\infty} \\
&\lesssim m^{-2} + ||\nabla v(t)||_{L^\infty} m^{-1}
\end{aligned} \tag{44}$$

For any strictly positive time $t > 0$,

$$\begin{aligned}
||\nabla v(t)||_{L^\infty} &\lesssim t^{-\frac{1}{2}}||\nabla v_0||_{L^d} + \int_0^t (t-s)^{-\frac{1}{2}}||u(s)||_{L^d}ds \\
&\lesssim t^{-\frac{1}{2}}||\nabla v_0||_{L^d} + ||u||_{L^2_t L^d} \lesssim t^{-\frac{1}{2}}||v_0||_{\dot{B}^{\frac{d}{p}}_{p,1}} + ||u||_{\mathscr{U}}
\end{aligned} \tag{45}$$

where we use embedding properties of Besov spaces:

$$\dot{B}^{\frac{d}{p}-1}_{p,1} \subset L^d, \quad p < d \tag{46}$$

By (44) and (45), for every fixed $\epsilon > 0$,

$$\frac{d}{dt}\int u(t,x)\phi_m(x)dx \to 0, \quad \text{uniformly in } t \in [\epsilon, \infty) \text{ as } m \to \infty \tag{47}$$

Since $u(t) \in L^1$,

$$\int u(x,t)\phi_m(x)dx \to \int u(t,x)dx \tag{48}$$

By (47) and (48),

$$\int u(t,x)dx = \int u(\epsilon,x)dx \ \text{ for all } t \in [\epsilon, \infty) \tag{49}$$

Since $u \in C([0,\infty); L^1)$, by letting $\epsilon \to 0$, we conclude that (40) holds. This completes the proof of theorem 1.3. ■

Remark 2 *(a) We can prove the continuity in time of u in L^1 by using the same argument in Chapter 3.*
(2) From the mass conservation of u, for $v_0 \in L^1$, v satisfies

$$\int v(t,x)dx = \int v_0(x)dx + t\int u_0(x)dx$$

4.2 Proof of Theorem 1.4

Finally, we prove that (u,v) is nonnegative if initial data are nonnegative. Suppose $u_0 \in \dot{B}^{\frac{d}{p}-2}_{p,1} \cap L^1 \cap L^2$. Let $u^-(t,x) = \min(u(t,x),0)$. We multiply the evolution equation of u by u^- and integrate over $\mathbb{R}^d$.

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}||u^-(t)||^2_{L^2} + \int_0^t ||\nabla u^-(s)||ds \\ \leq\ & \frac{1}{2}||u_0^-||^2_{L^2} + \int_0^t ||u^-(s)||_{L^2}||\nabla v(s)||_{L^\infty}||\nabla u^-(s)||_{L^2}ds \\ \leq\ & \frac{1}{2}||u_0^-||^2_{L^2} + \frac{1}{2}\int_0^t ||u^-(s)||^2_{L^2}||\nabla v||^2_{L^\infty}ds + \frac{1}{2}\int_0^t ||\nabla u^-(s)||^2_{L^2}ds \end{aligned} \tag{50}$$

By Gronwall's inequality,

$$||u^-||_{L^\infty_t L^2} \lesssim ||u_0^-||_{L^2}\exp(\int_0^\infty ||\nabla v(t)||^2_{L^\infty}dt) \tag{51}$$

which implies that $u(t,x) \geq 0$ for almost all $t > 0$ and $x \in \mathbb{R}^d$. Since the heat semigroup preserves the nonnegativity, $v(t,x) \geq 0$ for almost all $t > 0$ and $x \in \mathbb{R}^d$. This completes the proof of theorem 1.4. ■

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